A necessary and su cient condition for the non-trivial limit of the derivative martingale in a branching random walk

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Summary. We consider a branching random walk on the line. Biggins and Kyprianou [6] proved that, in the boundary case, the associated derivative martingale converges almost surly to a finite nonnegative limit, whose law serves as a fixed point of a smoothing transformation (Mandelbrot's cascade). In the present paper, we give a necessary and sufficient condition for the non-triviality of this limit and establish a Kesten-Stigum-like result.

Keywords. Branching random walk; derivative martingale; Mandelbrot's cascade; random walk conditioned to stay positive.

1 Introduction

We consider a discrete-time branching random walk (BRW) on the real line, which can be described in the following way. An initial ancestor, called the root and denoted by $\varnothing$, is created at the origin. It gives birth to some children which form the first generation and whose positions are given by a point process $\mathcal{L}$ on $\mathbb{R}$. For any integer $n \geq 1$, each individual in the nth generation gives birth independently of all others to its own children in the $(n+1)$th generation, and the displacements of its children from this individual's position is given by an independent copy of $\mathcal{L}$. The system goes on if there is no extinction. We thus obtain a genealogical tree, denoted by $\mathbb{T}$. For each vertex (individual) $u \in \mathbb{T}$, we denote its generation by $|u|$ and its position by $V(u)$. In particular, $V(\varnothing) = 0$ and $(V(u); |u| = 1) = \mathcal{L}$.

Note that the point process $\mathcal{L}$ plays the same role in the BRW as the offspring distribution in a Galton-Watson process. We introduce the Laplace-Stieltjes transform of $\mathcal{L}$ as follows:

$$\Phi(t) := \mathbf{E}\Big[\int_{\mathbb{R}} \mathrm{e}^{-tx}\mathcal{L}(\,\mathrm{d}x)\Big] = \mathbf{E}\Big[\sum_{|u|=1} \mathrm{e}^{-tV(u)}\Big], \text{ for } \forall t \in \mathbb{R}. \tag{1.1}$$

Let $\Psi(t) := \log \Phi(t)$. We always assume in this paper $\Psi(0) > 0$ so that $\mathbf{E}\Big[\sum_{|u|=1} 1\Big] > 1$. This yields that with strictly positive probability, the system survives. Let q be the probability of extinction. Clearly, $q < 1$.

Let $(\mathcal{F}_n; n \geq 0)$ be the natural filtration of this branching random walk, i.e. let $\mathcal{F}_n := \sigma\{(u, V(u)); |u| \leq n\}$. We introduce the additive martingale for any $t \in \mathbb{R}$,

$$W_n(t) := \sum_{|u|=n} \mathrm{e}^{-tV(u) - n\Psi(t)}. \tag{1.2}$$

It is a nonnegative martingale with respect to $(\mathcal{F}_n; n \geq 0)$, which converges almost surely to a finite nonnegative limit. Biggins [3] established a necessary and sufficient condition for the mean convergence of $W_n(t)$, and generalized Kesten-Stigum theorem for the Galton-Watson processes. A simpler proof based on a change of measures was given later by Lyons [14].

More generally, Biggins and Kyprianou [6] studied the martingales produced by the so-called mean-harmonic functions. Given suitable conditions on the offspring distribution $\mathcal{L}$ of the branching random walk, like the $X \log X$ condition of the Kesten-Stigum theorem, they gave a general treatment to obtain the mean convergence of these martingales. In this paper, following their ideas, we work on one special example and give a Kesten-Stigum-like theorem.

Throughout this paper, we consider the boundary case (in the sense of [7]) where $\Psi(1) = \Psi'(1) = 0$, i.e.,

$$\mathbf{E}\Big[\sum_{|u|=1} \mathrm{e}^{-V(u)}\Big] = 1, \qquad \mathbf{E}\Big[\sum_{|u|=1} V(u)\mathrm{e}^{-V(u)}\Big] = 0. \tag{1.3}$$

In addition, we assume that

$$\sigma^2 := \mathbf{E}\Big[\sum_{|u|=1} V(u)^2 \mathrm{e}^{-V(u)}\Big] \in (0, \infty). \tag{1.4}$$

We are interested in the derivative martingale, which is defined as follows:

$$D_n := \sum_{|u|=n} V(u)\mathrm{e}^{-V(u)}, \quad \forall n \geq 0. \tag{1.5}$$

It is a signed martingale with respect to $(\mathcal{F}_n)$, of mean zero. By Theorem 5.1 of [6], under (1.3) and (1.4), D_n converges almost surely to a finite nonnegative limit, denoted by D_∞. Moreover, D_∞ satisfies the following equation (Mandelbrot's cascade):

$$D_\infty = \sum_{|u|=1} \mathrm{e}^{-V(u)} D_\infty^{(u)}, \tag{1.6}$$

where $D_\infty^{(u)}$ are copies of D_∞ independent of each other and of $\mathcal{F}_1$. Note that D_∞ serves as a nonnegative fixed point of a smoothing transformation. From this point of view, the questions concerning the existence, uniqueness and asymptotic behavior of such fixed points have been much studied in the literature ([5, 7, 12, 13]). We are interested in the existence of a non-trivial fixed point, and we are going to determine when $\mathbf{P}(D_\infty > 0) > 0$.

It is known that $\mathbf{P}(D_\infty = 0)$ is equal to either the extinction probability q or 1 (see [1], for example). We say that the limit D_∞ is non-trivial if $\mathbf{P}(D_\infty > 0) > 0$, which means that $\mathbf{P}(D_\infty = 0) = q$. Otherwise, it is trivially zero. In this paper, we give a sufficient and necessary condition for the non-triviality of D_∞. The main result is stated as follows.

For any $y \in \mathbb{R}$, let $y_+ := \max\{y, 0\}$ and let $\log_+ y := \log(\max\{y, 1\})$. We introduce the following random variables:

$$Y := \sum_{|u|=1} \mathrm{e}^{-V(u)}, \quad Z := \sum_{|u|=1} V(u)_+ \mathrm{e}^{-V(u)}. \tag{1.7}$$

Theorem 1.1. *The limit of the derivative martingale D_n is non-trivial, namely $\mathbf{P}(D_\infty > 0) > 0$, if and only if the following condition holds:*

$$\mathbf{E}\Big(Z \log_+ Z + Y (\log_+ Y)^2\Big) < \infty. \tag{1.8}$$

Remark 1.2. *In [6], the authors studied the optimal condition for the non-triviality of D_∞. However, there is a small gap between the necessary condition and the sufficient condition for $\mathbf{P}(D_\infty > 0) > 0$ in their Theorem 5.2. Our result fills this gap and gives the analogue of the result of [15] in the case of branching Brownian motion.*

Remark 1.3. *Aïdékon proved that the condition (1.8) is sufficient for $\mathbf{P}(D_\infty > 0) > 0$ (see Proposition A.3 in the Appendix of [1]).*

The paper is organized as follows. Section 2 introduces a change of measures based on a truncated martingale which is closely related to the derivative martingale. We also prove a proposition concerning certain behaviors of a centered random walk conditioned to stay positive at the end of Section 2. Then, by using this proposition, we prove Theorem 1.1 in Section 3.

Throughout the paper, $(c_i)_{i\geq 0}$ denote positive constants. We write $\mathbf{E}[f;\, A]$ for $\mathbf{E}[f 1_A]$ and set $\sum_\emptyset := 0$.

2 Lyons' change of measures via truncated martingales

2.1 Truncated martingales

We begin with the well-known many-to-one lemma. For any $a \in \mathbb{R}$, let $\mathbf{P}_a$ be the probability measure such that $\mathbf{P}_a\Big(\big(V(u),\ u \in \mathbb{T}\big) \in \cdot\Big) = \mathbf{P}\Big(\big(V(u)+a,\ u \in \mathbb{T}\big) \in \cdot\Big)$. The corresponding expectation is denoted by $\mathbf{E}_a$. We write $\mathbf{P}$, $\mathbf{E}$ instead of $\mathbf{P}_0$, $\mathbf{E}_0$ for brevity. For any particle $u \in \mathbb{T}$, we denote by u_i its ancestor at the ith generation, for $0 \le i < |u|$. In addition, we write $u_{|u|} := u$. We thus denote its ancestral line by $[\![\varnothing,\ u]\!] := \{u_0,\ u_1, \cdots, u_{|u|}\}$.

Lemma 2.1 (Many-to-one)**.** *There exists a sequence of i.i.d centered random variables* $(S_{k+1} - S_k)$, $k \ge 0$ *such that for any* $n \ge 1$ *and any measurable function* $g : \mathbb{R}^n \to \mathbb{R}_+$, *we have*

$$\mathbf{E}_a\Big[\sum_{|u|=n} g\big(V(u_1), \cdots, V(u_n)\big)\Big] = \mathbf{E}_a\Big[\mathrm{e}^{S_n - a} g(S_1, \cdots, S_n)\Big], \tag{2.1}$$

with $\mathbf{P}_a[S_0 = a] = 1$.

In view of (1.4), $S_1 - S_0$ has a finite variance $\sigma^2 = \mathbf{E}[S_1^2] = \mathbf{E}[\sum_{|u|=1} V(u)^2 \mathrm{e}^{-V(u)}]$.

Let $U^-(\,\mathrm{d}y)$ be the renewal measure associated with the weak descending ladder height process of $(S_n,\ n \ge 0)$. Following the arguments in Section 2 of [4], we obtain that for any measurable function $f : \mathbb{R} \to \mathbb{R}_+$,

$$\mathbf{E}\Big[\sum_{j=0}^{\tau-1} f(-S_j)\Big] = \int_0^\infty f(y) U^-(\,\mathrm{d}y), \tag{2.2}$$

where τ be the first time that (S_n) enters $(0, \infty)$, namely $\tau := \inf\{k > 0,\ S_k \in (0, \infty)\}$ which is proper here. We define $R(x) := U^-([0, x))$ for all $x > 0$ and define $R(0) := 1$. Note that $R(x)$ equals the renewal function $U^-([0, x])$ at points of continuity. We collect the following properties of this function $R(x)$ which are consequences of the renewal theorem (see [4, 2, 17]).

Fact 2.2. *(i) There exists a positive constant* $c_0 > 0$ *such that*

$$\lim_{x \to \infty} \frac{R(x)}{x} = c_0. \tag{2.3}$$

(ii) There exist two constants $0 < c_1 < c_2 < \infty$ *such that*

$$c_1(1 + x) \le R(x) \le c_2(1 + x), \qquad \forall x \ge 0. \tag{2.4}$$

(iii) For any $x \geq 0$, we have $\mathbf{E}[R(S_1 + x)1_{(S_1+x>0)}] = R(x)$.

Let $\beta \geq 0$. Started from $V(\varnothing) = a$, we add a barrier at $-\beta$ to the branching random walk. Now, we define the following truncated random variables:

$$D_n^{(\beta)} := \sum_{|x|=n} R(V(x) + \beta)\mathrm{e}^{-V(x)} 1_{(\min_{1\leq k\leq n} V(x_k)>-\beta)}, \qquad \forall n \geq 1, \tag{2.5}$$

and $D_0^{(\beta)} := R(a + \beta)\mathrm{e}^{-a} 1_{(a\geq-\beta)}$.

Lemma 2.3. *For any $a \geq 0$ and $\beta \geq 0$, under $\mathbf{P}_a$, the process $(D_n^{(\beta)},\ n \geq 0)$ is a nonnegative martingale with respect to $(\mathcal{F}_n,\ n \geq 0)$.*

This lemma follows immediately from (iii) of Fact 2.2 and the branching property. We feel free to omit its proof and call $(D_n^{(\beta)})$ the truncated martingale. It also tells us that under $\mathbf{P}_a$, $(D_n^{(\beta)},\ n \geq 0)$ converges almost surely to a finite nonnegative limit, which we denote by $D_\infty^{(\beta)}$.

The connection between the limits of the derivative martingale and truncated martingales is recorded in the following Lemma, the proof of which can be referred to [6] and [1].

Lemma 2.4. *(1) If D_∞ is trivial, i.e., $\mathbf{P}(D_\infty = 0) = 1$, then for any $\beta \geq 0$, $D_\infty^{(\beta)}$ is trivially zero under $\mathbf{P}$.*

(2) Under $\mathbf{P}$, if there exists some $\beta \geq 0$ such that $D_\infty^{(\beta)}$ is trivially zero, so is D_∞.

Thanks to Lemma 2.4, we only need to investigate the truncated martingale $(D_n^{(0)}; n \geq 0)$ and determine when its limit is non-trivial.

2.2 Lyons' change of probabilities and spinal decomposition

Let $\beta = 0$. With this nonnegative martingale $(D_n^{(0)},\ n \geq 0)$, we define for any $a \geq 0$ a new probability measure $\mathbf{Q}_a$ such that for any $n \geq 1$,

$$\frac{d\mathbf{Q}_a}{d\mathbf{P}_a}\Big|_{\mathcal{F}_n} = \frac{D_n^{(0)}}{R(a)e^{-a}}. \tag{2.6}$$

$\mathbf{Q}_a$ is defined on $\mathcal{F}_\infty := \vee_{n\geq0}\mathcal{F}_n$. Let us give an intuitive description of the branching random walk under $\mathbf{Q}_a$, which is known as the spinal decomposition. We start from one single particle ω_0, located at the position $V(\omega_0) = a$. At time 1, it dies and produces a point process distributed as $(V(u);\ |u| = 1)$ under $\mathbf{Q}_a$. Among the children of ω_0, ω_1 is chosen to be u with probability proportional to $R(V(u))\mathrm{e}^{-V(u)}1_{(V(u)>0)}$. At each time $n+1$, each particle v in the nth generation dies and produces independently a point process distributed as $(V(u);\ |u|=1)$ under $\mathbf{P}_{V(v)}$ except ω_n, which dies and generates independently a point process distributed as $(V(u);\ |u|=1)$ under $\mathbf{Q}_{V(\omega_n)}$. And then ω_{n+1} is chosen to be u among the children of ω_n, with probability proportional to $R(V(u))\mathrm{e}^{-V(u)}1_{(\min_{1\le k\le n+1} V(u_k)>0)}$. We still use $\mathbb{T}$ to denote the genealogical tree. Then $(\omega_n;\ n\ge 0)$ is an infinite ray in $\mathbb{T}$, which is called the spine. The rigorous proof was given in Appendix A of [1]. Indeed, this type of measures' change and the establishment of a spinal decomposition have been developed in various cases of the branching framework; see, for example [14, 11, 8, 10].

We state the following fact about the distribution of the spine process $(V(\omega_n);\ n\ge 0)$ under $\mathbf{Q}_a$.

Fact 2.5. *Let $a\ge 0$. For any $n\ge 0$ and any measurable function $g:\mathbb{R}^{n+1}\to\mathbb{R}_+$, we have*

$$\mathbf{E}_{\mathbf{Q}_a}\Big[g(V(\omega_0),\cdots,V(\omega_n))\Big]=\frac{1}{R(a)}\mathbf{E}_a\Big[g(S_0,\cdots,S_n)R(S_n);\ \min_{1\le k\le n} S_i>0\Big], \tag{2.7}$$

where (S_n) is the same as that in Lemma 2.1.

For convenience, let $(\zeta_n;\ n\ge 0)$ be a stochastic process under $\mathbf{P}_a$ such that

$$\mathbf{P}_a[\Big(\zeta_n;\ n\ge 0\Big)\in\cdot]=\mathbf{Q}_a[\Big(V(\omega_n);\ n\ge 0\Big)\in\cdot]. \tag{2.8}$$

Obviously, under $\mathbf{P}_a$, $(\zeta_n;\ n\ge 0)$ is a Markov chain with transition probabilities P so that, for any $x\ge 0$, $P(x,\,\mathrm{d}y)=\frac{R(y)}{R(x)}1_{(y>0)}\mathbf{P}_x(S_1\in\,\mathrm{d}y)$. This process (ζ_n) is usually called a random walk conditioned to stay positive. It has been arisen and studied in, for instance, [17, 2, 4, 18]. In what follows, we state some results about (ζ_n), which will be useful later in Section 3.

2.3 Random walk conditioned to stay positive

Recall that (S_n) is a centered random walk on $\mathbb{R}$ with finite variance σ^2. Let τ_- be the first time that (S_n) hits $(-\infty,0]$, namely, $\tau_-:=\inf\{k\ge 1:\ S_k\le 0\}$. Let $(T_k,\,H_k;\ k\ge 0)$ be the strict ascending ladder epochs and heights of $(S_n;\ n\ge 0)$, i.e., $T_0=0$, $H_0:=S_0$ and for any $k\ge 1$, $T_k:=\inf\{j>T_{k-1}:\ S_j>H_{k-1}\}$, $H_k:=S_{T_k}$. We denote by $U(\,\mathrm{d}x)$ the corresponding renewal measure (see Chapter XII in [9], for example). Then, similarly to (2.2), for any measurable function $f:\mathbb{R}\to\mathbb{R}_+$,

$$\mathbf{E}\Big[\sum_{n=0}^{\tau_- -1} f(S_n)\Big]=\mathbf{E}\Big[\sum_{k\ge 0} f(H_k)\Big]=\int_0^\infty f(x)U(\,\mathrm{d}x). \tag{2.9}$$

We deduce from (2.7) and (2.9) that

$$\mathbf{E}\Big[\sum_{n\geq 0} f(\zeta_n)\Big] = \mathbf{E}_{\mathbf{Q}_0}\Big[\sum_{n\geq 0} f(V(\omega_n))\Big] = \sum_{n\geq 0}\mathbf{E}\Big[f(S_n)R(S_n)1_{(\min_{1\leq k\leq n} S_k>0)}\Big]$$
$$= \mathbf{E}\Big[\sum_{n=0}^{\tau_- -1} f(S_n)R(S_n)\Big] = \int_0^\infty f(x)R(x)U(\,\mathrm{d}x). \tag{2.10}$$

Recall also that $U^-(\,\mathrm{d}x)$ is the renewal measure associated with the weak descending ladder height process of (S_n). By the renewal theorem (see P.360 in [9]), there exist two constants c_3, $c_4 > 0$ such that for $\forall x,\ y\geq 0$,

$$c_3(1+x)\leq U([0,x])\leq c_4(1+x), \qquad 0\leq U([x,x+y])\leq c_4(1+y); \tag{2.11}$$
$$c_3(1+x)\leq U^-([0,x])\leq c_4(1+x), \qquad 0\leq U^-([x,x+y])\leq c_4(1+y). \tag{2.12}$$

Given a non-increasing function $F\geq 0$, we present the following proposition, which gives a necessary and sufficient condition for the infinity of the series $\sum_n F(\zeta_n)$.

Proposition 2.6. *Let $F:[0,\infty)\to[0,\infty)$ be non-increasing. Then*

$$\int_0^\infty F(y)y\,\mathrm{d}y = \infty \Longleftrightarrow \sum_{n\geq 0} F(\zeta_n) = \infty, \quad \mathbf{P}\text{-}a.s. \tag{2.13}$$

Note that (ζ_n) can be viewed as a discrete-time counterpart of the 3-dimensional Bessel process, for which a similar result holds (see, for instance, Ex 2.5, Chapter XI of [16]). And we will prove (2.13) in a similar way as for the Bessel process.

Proof. Observe that $0\leq F(x)\leq F(0)<\infty$ for any $x\geq 0$. So there is no difference between the two events $\{\sum_{n\geq 0} F(\zeta_n)=\infty\}$ and $\{\sum_{n\geq 1} F(\zeta_n)=\infty\}$.

We first prove "$\Longleftarrow$" in (2.13). It follows from (2.4) and (2.11) that

$$\int_0^\infty F(y)y\,\mathrm{d}y = \infty \Longleftrightarrow \int_0^\infty F(y)R(y)U(\,\mathrm{d}y) = \infty. \tag{2.14}$$

Actually, by (2.10),

$$\mathbf{E}\Big[\sum_{n\geq 0} F(\zeta_n)\Big] = \int_0^\infty F(y)R(y)U(\,\mathrm{d}y).$$

Clearly, $\mathbf{P}\Big[\sum_{n\geq 0} F(\zeta_n)=\infty\Big]=1$ yields $\int_0^\infty F(y)R(y)U(\,\mathrm{d}y)=\infty$. The "$\Longleftarrow$" in (2.13) is hence proved.

To prove "$\Longrightarrow$" in (2.13), we only need to show that if $\mathbf{P}\Big[\sum_{n\geq 0} F(\zeta_n) = \infty\Big] < 1$, then $\int_0^\infty F(y) y\,\mathrm{d}y < \infty$. From now on, we suppose that $\mathbf{P}\Big[\sum_{n\geq 0} F(\zeta_n) = \infty\Big] < 1$, which is equivalent to say that,

$$\mathbf{P}\Big[\sum_{n\geq 1} F(\zeta_n) < \infty\Big] > 0. \tag{2.15}$$

We draw support from Tanaka's construction for the random walk conditioned to stay positive ([17, 4]). Recall that $\tau = \inf\{k \geq 1 : S_k \in (0,\infty)\}$. We hence obtain an excursion $(S_j;\ 0 \leq j \leq \tau)$, which is denoted by $\xi = (\xi(j),\ 0 \leq j \leq \tau)$. Let $\{\xi_k = (\xi_k(j), 0 \leq j \leq \tau_k);\ k \geq 1\}$ be a sequence of independent copies of ξ. For any $k \geq 1$, let

$$\nu_k(j) := \xi_k(\tau_k) - \xi_k(\tau_k - j),\ \forall 0 \leq j \leq \tau_k. \tag{2.16}$$

This brings out another sequence of i.i.d. excursions $\{\nu_k = (\nu_k(j),\ 0 \leq j \leq \tau_k);\ k \geq 1\}$, based on which we reconstruct the random walk conditioned to stay position (ζ_n) in the following way. Define for any $k \geq 1$,

$$T_k^+ := \tau_1 + ... + \tau_k; \tag{2.17}$$

$$H_k^+ := \nu_1(\tau_1) + ... + \nu_k(\tau_k) = \xi_1(\tau_1) + ... + \xi_k(\tau_k), \tag{2.18}$$

and let $T_0^+ = H_0^+ = 0$. Then the process

$$\zeta_n = H_k^+ + \nu_{k+1}(n - T_k^+), \quad \text{for } T_k^+ < n \leq T_{k+1}^+, \tag{2.19}$$

with $\zeta_0 = 0$, is what we need.

We actually establish un process distributed as (ζ_n). For brevity, we still denote it by (ζ_n) without changing any conclusion in this proof. For any $k \geq 1$, let

$$\chi_k(F) := \sum_{n=T_{k-1}^+ + 1}^{T_k^+} F(\zeta_n) = \sum_{j=1}^{\tau_k} F\Big(H_{k-1}^+ + \nu_k(j)\Big), \tag{2.20}$$

so that $\sum_{n\geq 1} F(\zeta_n) = \sum_{k\geq 1} \chi_k(F)$.

By (2.16), we get that

$$\begin{aligned} \chi_k(F) &= \sum_{j=1}^{\tau_k} F\Big(H_{k-1}^+ + \xi_k(\tau_k) - \xi_k(\tau_k - j)\Big) \\ &= \sum_{j=0}^{\tau_k - 1} F\Big(H_k^+ - \xi_k(j)\Big). \end{aligned}$$

(2.15) hence becomes that

$$\mathbf{P}\Big[\sum_{k\ge 1}\chi_k(F)<\infty\Big]=\mathbf{P}\Big[\sum_{k\ge 1}\sum_{j=0}^{\tau_k-1}F\Big(H_k^+-\xi_k(j)\Big)<\infty\Big]>0. \tag{2.21}$$

By Theorem 1 in Chapter XVIII.5 of [9], as (S_n) is of finite variance, we have $b^+ := \mathbf{E}[H_1^+] = \mathbf{E}[S_\tau] < \infty$. It follows from Strong Law of Large Numbers that $\mathbf{P}$-a.s.,

$$\lim_{k\to\infty}\frac{H_k^+}{k}=b^+. \tag{2.22}$$

Let $A > \max\{1, b^+\}$. This tells us that $\mathbf{P}$-a.s., for all large k, $H_k^+ \le Ak$. As F is non-increasing, one sees that

$$\mathbf{P}\Big[\sum_{k\ge 1}\sum_{j=0}^{\tau_k-1}F\Big(Ak-\xi_k(j)\Big)<\infty\Big]\ge\mathbf{P}\Big[\sum_{k\ge 1}\sum_{j=0}^{\tau_k-1}F\Big(H_k^+-\xi_k(j)\Big)<\infty\Big]>0. \tag{2.23}$$

For any $k\ge 1$ let

$$\widetilde{\chi}_k := \sum_{j=0}^{\tau_k-1}F\Big(Ak-\xi_k(j)\Big). \tag{2.24}$$

So, $\mathbf{P}\Big[\sum_{k\ge 1}\widetilde{\chi}_k<\infty\Big]>0$. Recall that $\{\xi_k, k\ge 1\}$ is a sequence of independent copies of $(S_j;\ 0\le j\le\tau)$. This yields the independence of the sequence $\{\widetilde{\chi}_k, k\ge 1\}$. It follows from Kolmogorov's 0-1 law that

$$\mathbf{P}\Big[\sum_{k\ge 1}\sum_{j=0}^{\tau_k-1}F(Ak-\xi_k(j))<\infty\Big]=\mathbf{P}\Big[\sum_{k\ge 1}\widetilde{\chi}_k<\infty\Big]=1. \tag{2.25}$$

Moreover, let $E_M := \Big\{\sum_{k\ge 1}\widetilde{\chi}_k < M\Big\}$ for any $M>0$. Either there exists some $M_0<\infty$ such that $\mathbf{P}[E_{M_0}]=1$, or $\mathbf{P}[E_M]<1$ for all $M\in(0,\infty)$. On the one hand, if $\mathbf{P}[E_{M_0}]=1$ for some $M_0<\infty$, then

$$\begin{aligned} M_0 &\ge \mathbf{E}\Big[\sum_{k\ge 1}\widetilde{\chi}_k\Big]=\mathbf{E}\Big[\sum_{k\ge 1}\sum_{j=0}^{\tau_k-1}F\Big(Ak-\xi_k(j)\Big)\Big]\\ &= \sum_{k\ge 1}\mathbf{E}\Big[\sum_{j=0}^{\tau-1}F(Ak-S_j)\Big]\\ &= \sum_{k\ge 1}\int_0^\infty F(Ak+y)U^-(dy), \end{aligned}$$

where the last equality follows from (2.9). One sees that $\sum_{k\geq 1}\int_0^\infty F(Ak+y)U^-(\,\mathrm{d}y) < \infty$. It follows from the renewal theorem that there exists $B>0$ such that $U^-([jB, jB+B)) > \delta > 0$ for any $j \geq 0$. As F is non-increasing,

$$\sum_{k\geq 1}\sum_{j\geq 1} F(Ak+Bj)\delta \leq \sum_{k\geq 1}\int_0^\infty F(Ak+y)U^-(\,\mathrm{d}y) < \infty. \tag{2.26}$$

We hence observe that $\int_A^\infty \mathrm{d}z \int_B^\infty F(y+z)\,\mathrm{d}y \leq \sum_{k\geq 1}\sum_{j\geq 1} F(Ak+Bj)AB < \infty$. This implies that

$$\int_0^\infty F(x)x\,\mathrm{d}x = \int_0^\infty dz \int_0^\infty F(z+y)\,\mathrm{d}y \leq F(0)AB + \int_A^\infty \mathrm{d}z\int_B^\infty F(y+z)\,\mathrm{d}y < \infty,$$

which is what we need.

On the other hand, if $\mathbf{P}[E_M] < 1$ for all $M \in (0,\infty)$, we have $\lim_{M\uparrow\infty}\mathbf{P}[E_M] = 1$ because of (2.25). For any $k\geq 1$ and any $\ell \geq 1$, define:

$$\Lambda_\ell^{(k)} := \sum_{j=0}^{\tau_k-1} 1_{\{A(\ell-1)\leq -\xi_k(j) < A\ell\}}. \tag{2.27}$$

As $\sum_{\ell\geq 1} 1_{\{A(\ell-1)\leq -\xi_k(j)<A\ell\}} = 1$, we get that for any $k\geq 1$,

$$\begin{aligned}
\widetilde{\chi}_k &= \sum_{j=0}^{\tau_k-1} F\Big(Ak-\xi_k(j)\Big)\sum_{\ell\geq 1} 1_{\{A(\ell-1)\leq -\xi_k(j)<A\ell\}} \\
&= \sum_{\ell\geq 1}\sum_{j=0}^{\tau_k-1} F\Big(Ak-\xi_k(j)\Big) 1_{\{A(\ell-1)\leq -\xi_k(j)<A\ell\}} \\
&\geq \sum_{\ell\geq 1} F(Ak+A\ell)\Lambda_\ell^{(k)},
\end{aligned}$$

where the last inequality holds because F is non-increasing. It follows that

$$\begin{aligned}
\sum_{k\geq 1}\widetilde{\chi}_k &\geq \sum_{k\geq 1}\sum_{\ell\geq 1} F(Ak+A\ell)\Lambda_\ell^{(k)} = \sum_{n=2}^\infty F(An)\sum_{k=1}^{n-1}\Lambda_{n-k}^{(k)} \\
&= \sum_{m=1}^\infty F(Am+A)mY_m,
\end{aligned} \tag{2.28}$$

where

$$Y_m := \frac{\sum_{k=1}^m \Lambda_{m+1-k}^{(k)}}{m},\ \forall m\geq 1. \tag{2.29}$$

We claim that there exists a $M_1 > 0$ sufficiently large such that for any $m \geq 1$,

$$c_6 \geq \mathbf{E}\Big[Y_m 1_{E_{M_1}}\Big] \geq c_5 > 0, \tag{2.30}$$

where c_5, c_6 are positive constants. We postpone the proof of (2.30) and go back to (2.28). It follows that

$$\begin{aligned} M_1 &\geq \mathbf{E}\Big[1_{E_{M_1}} \sum_{k\geq 1} \widetilde{\chi}_k\Big] \geq \mathbf{E}\Big[1_{E_{M_1}} \sum_{m=1}^{\infty} F(Am+A)mY_m\Big] \\ &\geq \sum_{m\geq 1} F(Am+A)m\mathbf{E}\Big[Y_m 1_{E_{M_1}}\Big]. \end{aligned} \tag{2.31}$$

By (2.30), we obtain that

$$\sum_{m\geq 1} F(Am+A)m \leq M_1/c_5 < \infty. \tag{2.32}$$

This implies that $\int_0^\infty F(y)y\,\mathrm{d}y < \infty$ thus completes the proof of Proposition 2.6.

It remains to prove (2.30).

We begin with the first and second moments of Y_m. Since $\{\omega_k; k \geq 1\}$ are i.i.d. copies of $(S_j, 0 \leq j \leq \tau)$, $(\Lambda_\ell^{(k)}; \ell \geq 1)$, $k \geq 1$ are i.i.d. This yields that

$$\begin{aligned} \mathbf{E}[Y_m] &= \frac{1}{m}\sum_{k=1}^{m} \mathbf{E}[\Lambda^{(k)}_{m+1-k}] = \frac{1}{m}\sum_{k=1}^{m} \mathbf{E}[\Lambda^{(1)}_{m+1-k}] \\ &= \frac{1}{m}\mathbf{E}\Big[\sum_{k=1}^{m} \Lambda_k^{(1)}\Big] = \frac{1}{m}\mathbf{E}\Big[\sum_{j=0}^{\tau-1} 1_{\{-S_j<Am\}}\Big] \\ &= \frac{R(Am)}{m}. \end{aligned} \tag{2.33}$$

where the last equality comes from (2.2). By (2.4), for any $m \geq 1$,

$$c_1 A \leq \mathbf{E}[Y_m] \leq c_2(A+1) =: c_6. \tag{2.34}$$

Obviously, we have $\mathbf{E}[Y_m 1_{E_M}] \leq c_6$ for any $m \geq 1$ and any $M > 0$. The fact that $\Lambda^{(k)}_{\cdot}, k \geq 1$, are i.i.d. yields also that

$$\mathrm{Var}(Y_m) = \frac{1}{m^2}\sum_{k=1}^{m} \mathrm{Var}(\Lambda_k^{(1)}) \leq \frac{1}{m^2}\sum_{k=1}^{m} \mathbf{E}[(\Lambda_k^{(1)})^2]. \tag{2.35}$$

Note that $\Lambda_1^{(1)}$ is distributed as $\sum_{j=0}^{\tau-1} 1_{\{-S_j<A\}}$ with $\tau = \inf\{k>0 : S_k>0\}$. We see that

$$\begin{aligned}\mathbf{E}\Big[\Big(\Lambda_1^{(1)}\Big)^2\Big] &= \mathbf{E}\Big[\Big(\sum_{j=0}^{\tau-1} 1_{\{-S_j<A\}}\Big)^2\Big] \\ &\le 2\mathbf{E}\Big[\sum_{j=0}^{\tau-1} 1_{\{-S_j<A\}} \sum_{k=j}^{\tau-1} 1_{\{-S_k<A\}}\Big].\end{aligned}$$

By Markov property, we obtain that

$$\mathbf{E}\Big[\Big(\Lambda_1^{(1)}\Big)^2\Big] \le 2\mathbf{E}\Big[\sum_{j=0}^{\tau-1} 1_{\{-S_j<A\}} R(A,-S_j)\Big], \tag{2.36}$$

where

$$R(x,y) := \mathbf{E}\Big[\sum_{i=0}^{\tau_y-1} 1_{\{S_i>y-x\}}\Big] \text{ with } \tau_y := \inf\{k>0 :\ S_k>y\} \text{ for } x,y\ge 0. \tag{2.37}$$

It follows from (2.2) that

$$\mathbf{E}\Big[\Big(\Lambda_1^{(1)}\Big)^2\Big] \le 2\int_0^A R(A,y)U^-(\,\mathrm{d}y). \tag{2.38}$$

Consider now the strict ascending ladder epochs and heights $(T_k,\ H_k)$ of (S_n). We get that

$$R(x,y) = \mathbf{E}\Big[\sum_{k=0}^{\infty} 1_{\{y\ge H_k>y-x\}} \sum_{n=T_k}^{T_{k+1}-1} 1_{\{S_n>y-x\}}\Big].$$

By applying the Markov property at the times $(T_k;\ k\ge 1)$ and (2.2), we have for $x,\ y\ge 0$,

$$R(x,y) = \mathbf{E}\Big[\sum_{k\ge 0} R(H_k+x-y) 1_{\{y\ge H_k>y-x\}}\Big] = \int_{(y-x)_+}^{y} R(x-y+z)U(\,\mathrm{d}z). \tag{2.39}$$

Plugging it into (2.38) then using (2.4), (2.12) and (2.11) implies that

$$\mathbf{E}\Big[\Big(\Lambda_1^{(1)}\Big)^2\Big] \le c_7(1+A)^3 \le c_8 A^3, \tag{2.40}$$

(see also Lemma 2 in [4]).

Moreover, for any $\ell\ge 2$, $\Lambda_\ell^{(1)}$ has the same law as $\sum_{j=0}^{\tau-1} 1_{\{\ell A-A\le -S_j<\ell A\}}$. Similarly, we get that

$$\begin{aligned}\mathbf{E}\Big[\Big(\Lambda_\ell^{(1)}\Big)^2\Big] &= \mathbf{E}\Big[\Big(\sum_{j=0}^{\tau-1} 1_{\{\ell A-A\le -S_j<\ell A\}}\Big)^2\Big] \\ &\le 2\mathbf{E}\Big[\sum_{j=0}^{\tau-1} 1_{\{\ell A-A\le -S_j<\ell A\}} \sum_{k=j}^{\tau-1} 1_{\{\ell A-A\le -S_k<\ell A\}}\Big].\end{aligned}$$

Once again, by Markov property then by (2.2),

$$\mathbf{E}\Big[\Big(\Lambda_\ell^{(1)}\Big)^2\Big] \le 2\mathbf{E}\Big[\sum_{j=0}^{\tau-1} 1_{\{\ell A - A \le -S_j < \ell A\}}\Big(R(\ell A, -S_j) - R(\ell A - A, -S_j)\Big)\Big]$$
$$= 2\int_{\ell A - A}^{\ell A}\Big(R(\ell A, y) - R(\ell A - A, y)\Big)U^-(\,\mathrm{d}y).$$

Plugging (2.39) into it yields that for $\ell \ge 2$,

$$\begin{aligned}\mathbf{E}\Big[\Big(\Lambda_\ell^{(1)}\Big)^2\Big] &\le 2\int_{\ell A - A}^{\ell A}\Big(\int_0^y R(\ell A - y + z)U(\,\mathrm{d}z) - \int_{y-\ell A + A}^{y} R(\ell A - A - y + z)U(\,\mathrm{d}z)\Big)U^-(\,\mathrm{d}y)\\ &= 2\int_{\ell A - A}^{\ell A}\Big(\int_0^{y-\ell A + A} R(\ell A - y + z)U(\,\mathrm{d}z)\\ &\qquad + \int_{y-\ell A + A}^{y} U^-([\ell A - A - y + z, \ell A - y + z))U(\,\mathrm{d}z)\Big)U^-(\,\mathrm{d}y),\end{aligned}$$

where the last equality holds because $R(x) = U^-([0,x))$. Observe that $R(\ell A - y + z) \le R(A)$ for $0 \le z \le y - \ell A + A$ and $\ell A - A \le y \le \ell A$. Recall that $A \ge 1$. By (2.4), (2.11) and (2.12),

$$\begin{aligned}\mathbf{E}\Big[\Big(\Lambda_\ell^{(1)}\Big)^2\Big] &\le c_9\int_{\ell A - A}^{\ell A}\Big(\int_0^{y-\ell A + A}(A+1)U(\,\mathrm{d}z) + \int_{y-\ell A + A}^{y}(1+A)U(\,\mathrm{d}z)\Big)U^-(\,\mathrm{d}y)\\ &\le c_{10}(A+1)\int_{\ell A - A}^{\ell A}\Big(y+1\Big)U^-(\,\mathrm{d}y)\\ &\le c_{11}\ell A^3,\end{aligned}$$

with $c_{11} \ge c_8$. Going back to (2.35), for any $m \ge 1$,

$$\mathrm{Var}(Y_m) \le \frac{\sum_{\ell=1}^m c_{11}\ell A^3}{m^2} \le c_{12}A^3. \tag{2.41}$$

Combining this with (2.34) implies that $\mathbf{E}[Y_m^2] = \mathrm{Var}(Y_m) + \mathbf{E}[Y_m]^2 \le c_2^2(1+A)^2 + c_{12}A^3$. We then use Paley-Zygmund inequality to obtain that

$$\mathbf{P}\Big[Y_m > \frac{1}{2}\mathbf{E}[Y_m]\Big] \ge \frac{\mathbf{E}[Y_m]^2}{4\mathbf{E}[Y_m^2]} \ge \frac{c_1^2 A^2}{4(c_2^2(1+A)^2 + c_{12}A^3)} := c_{13} > 0. \tag{2.42}$$

So for any $0 \le u \le c_1 A/2 \le \mathbf{E}[Y_m]/2$, we have

$$\mathbf{P}\Big(Y_m \le u\Big) \le \mathbf{P}\Big(Y_m \le \mathbf{E}[Y_m]/2\Big) \le 1 - c_{13}. \tag{2.43}$$

There exists $M_1 > 0$ such that $\mathbf{P}(E_{M_1}) \geq 1 - c_{13}/2$, since $\lim_{M\uparrow\infty} \mathbf{P}[E_M] = 1$. For such $M_1 > 0$,

$$\mathbf{E}[Y_m 1_{E_{M_1}}] = \mathbf{E}\Big[\int_0^{Y_m} 1_{E_{M_1}} du\Big] = \int_0^\infty \mathbf{P}[\{Y_m > u\} \cap E_{M_1}] du. \tag{2.44}$$

Notice that $\mathbf{P}[\{Y_m > u\} \cap E_{M_1}] \geq \Big(\mathbf{P}[E_{M_1}] - \mathbf{P}[Y_m \leq u]\Big)_+$, which is larger than $c_{13}/2$ when $0 \leq u \leq c_1 A/2$. As a consequence,

$$\mathbf{E}[Y_m 1_{E_{M_1}}] = \int_0^\infty \mathbf{P}[\{Y_m > u\} \cap E_{M_1}] du \geq \int_0^{c_1 A/2} \frac{c_{13}}{2} du = \frac{c_1 c_{13} A}{4} =: c_5 > 0. \tag{2.45}$$

This completes the proof of (2.30), hence completes the proof of "$\Longrightarrow$" in (2.13). Proposition 2.6 is proved. □

3 Proof of the main theorem

Recall that we are in the regime that

$$\mathbf{E}\Big[\sum_{|u|=1} \mathrm{e}^{-V(u)}\Big] = 1, \quad \mathbf{E}\Big[\sum_{|u|=1} V(u)\mathrm{e}^{-V(u)}\Big] = 0, \quad \sigma^2 = \mathbf{E}\Big[\sum_{|u|=1} V(u)^2 \mathrm{e}^{-V(u)}\Big] < \infty. \tag{3.1}$$

Recall also that equivalence in Theorem 1.1 is as follows:

$$\mathbf{E}\Big[Y\Big(\log_+ Y\Big)^2\Big] + \mathbf{E}\Big[Z \log_+ Z\Big] < \infty \Longleftrightarrow \mathbf{P}[D_\infty > 0] > 0. \tag{3.2}$$

with $Y = \sum_{|u|=1} \mathrm{e}^{-V(u)}$ and $Z = \sum_{|u|=1} V(u)_+ \mathrm{e}^{-V(u)}$.

This section is devoted to proving that the condition on the left-hand side of (3.2) (i.e. (1.8)) is necessary and sufficient for mean convergence of the truncated martingale $\Big\{D_n^{(0)} = \sum_{|u|=n} R(V(u))\mathrm{e}^{-V(u)} 1_{\{V(u_k)>0, \forall 1\leq k\leq n\}}; n \geq 0\Big\}$. In view of Lemma 2.4, this follows the non-triviality of D_∞, hence proves Theorem 1.1.

In what follows, we state a result about the mean convergence of the truncated martingale $\Big\{D_n^{(0)}; n \geq 0\Big\}$, which is one special case of Theorem 2.1 in Biggins and Kyprianou [6].

Define

$$X := \frac{D_1^{(0)}}{D_0^{(0)}} 1_{(D_0^{(0)}>0)} + 1_{(D_0^{(0)}=0)}. \tag{3.3}$$

Then for any $a \geq 0$, under $\mathbf{P}_a$, $X = \frac{\sum_{|u|=1} R(V(u))\mathrm{e}^{-V(u)} 1_{(V(u)>0)}}{R(a)\mathrm{e}^{-a}}$.

Theorem 3.1 (Biggins and Kyprianou [6]). *(ζ_n) is a random walk conditioned to stay positive, whose law was given in (2.8).*

(i) If

$$\mathbf{P}\text{-}a.s.\ \sum_{n\geq 1}\mathbf{E}_{\zeta_n}\Big[X\Big(R(\zeta_n)\mathrm{e}^{-\zeta_n}X\wedge 1\Big)\Big]<\infty, \tag{3.4}$$

then $\mathbf{E}[D_\infty^{(0)}]=R(0)$.

(ii) If for all $y>0$,

$$\mathbf{P}\text{-}a.s.\ \sum_{n=1}^{\infty}\mathbf{E}_{\zeta_n}\Big[X;\ R(\zeta_n)\mathrm{e}^{-\zeta_n}X\geq y\Big]=\infty, \tag{3.5}$$

then $\mathbf{E}[D_\infty^{(0)}]=0$.

Our proof relies on this theorem. First, in Subsection 3.1, we give a short proof for the sufficient part to accomplish our arguments even though it has already been proved in [1]. In Subsection 3.2, we prove that (1.8) is also the necessary condition by using Proposition 2.6.

3.1 (1.8) is a sufficient condition

This subsection is devoted to proving that

$$\mathbf{E}\Big[Y\Big(\log_+ Y\Big)^2\Big]+\mathbf{E}\Big[Z\log_+ Z\Big]<\infty\Longrightarrow\mathbf{E}[D_\infty^{(0)}]=R(0)=1. \tag{3.6}$$

Proof of (3.6). According to (i) of Theorem 3.1, it suffices to show that

$$\mathbf{E}\Big[Y\Big(\log_+ Y\Big)^2\Big]+\mathbf{E}\Big[Z\log_+ Z\Big]<\infty\Longrightarrow\mathbf{P}\text{-a.s.}\ \sum_{n\geq 0}\mathbf{E}_{\zeta_n}\Big[X\Big(R(\zeta_n)\mathrm{e}^{-\zeta_n}X\wedge 1\Big)\Big]<\infty. \tag{3.7}$$

For any particle $x\in\mathbb{T}\setminus\{\varnothing\}$, we denote its parent by $\overleftarrow{u}$ and define its relative displacement by

$$\Delta V(u):=V(u)-V(\overleftarrow{u}). \tag{3.8}$$

Then for any $a\in\mathbb{R}$, under $\mathbf{P}_a$, $(\Delta V(u);\ |u|=1)$ is distributed as $\mathcal{L}$. Let $\widetilde{Y}:=\sum_{|u|=1}\mathrm{e}^{-\Delta V(u)}$ and $\widetilde{Z}:=\sum_{|u|=1}\Big(\Delta V(u)\Big)_+\mathrm{e}^{-\Delta V(u)}$ so that $\mathbf{P}_a\Big[\Big(\widetilde{Y},\,\widetilde{Z}\Big)\in\cdot\Big]=\mathbf{P}[(Y,\,Z)\in\cdot]$.

Note that under $\mathbf{P}_{\zeta_n}$,

$$X = \frac{\sum_{|u|=1} R(V(u))\mathrm{e}^{-V(u)} 1_{(V(u)>0)}}{R(\zeta_n)e^{-\zeta_n}}$$

$$(3.9) \qquad = \frac{\sum_{|u|=1} R(\zeta_n + \Delta V(u))\mathrm{e}^{-\Delta V(u)} 1_{(\Delta V(u)>-\zeta_n)}}{R(\zeta_n)},$$

where $(\Delta V(u);\ |u|=1)$ is independent of ζ_n. By (2.4), it follows that

$$\begin{aligned} X &\leq \frac{\sum_{|u|=1} c_2(\zeta_n+1)\mathrm{e}^{-\Delta V(u)} 1_{(\Delta V(u)>-\zeta_n)}}{R(\zeta_n)} + \frac{\sum_{|u|=1} c_2 \Delta V(u)\mathrm{e}^{-\Delta V(u)} 1_{(\Delta V(u)>-\zeta_n)}}{R(\zeta_n)} \\ &\leq \frac{c_2}{c_1}\sum_{|u|=1} \mathrm{e}^{-\Delta V(u)} + c_2\frac{\sum_{|u|=1} \Delta V(u)_+ \mathrm{e}^{-\Delta V(u)}}{R(\zeta_n)} \\ &\leq c_{14}\Big(\widetilde{Y} + \frac{\widetilde{Z}}{R(\zeta_n)}\Big) \leq 2c_{14}\max\Big\{\widetilde{Y},\ \frac{\widetilde{Z}}{R(\zeta_n)}\Big\}, \end{aligned}$$

where $\Big(\widetilde{Y},\ \widetilde{Z}\Big)$ is independent of ζ_n. This implies that

$$\begin{aligned} &\sum_{n\geq 1} \mathbf{E}_{\zeta_n}\Big[X\Big(R(\zeta_n)\mathrm{e}^{-\zeta_n} X \wedge 1\Big)\Big] \\ \leq\ & c_{15}\bigg(\sum_{n\geq 0}\mathbf{E}\Big[\widetilde{Y}\Big(R(\zeta_n)\mathrm{e}^{-\zeta_n}\widetilde{Y}\wedge 1\Big)\Big|\zeta_n\Big] + \sum_{n\geq 0}\frac{1}{R(\zeta_n)}\mathbf{E}\Big[\widetilde{Z}\Big(\mathrm{e}^{-\zeta_n}\widetilde{Z}\wedge 1\Big)\Big|\zeta_n\Big]\bigg) \\ (3.10) \qquad =:\ & c_{15}\Big(\Sigma_1 + \Sigma_2\Big). \end{aligned}$$

Hence we only need to prove that

$$(3.11) \qquad \mathbf{E}\Big[Y\Big(\log_+ Y\Big)^2\Big] + \mathbf{E}\Big[Z\log_+ Z\Big] < \infty \Longrightarrow \mathbf{E}\Big[\Sigma_1\Big] + \mathbf{E}\Big[\Sigma_2\Big] < \infty,$$

which leads to (3.7). On the one hand, as (2.4) gives that $R(x) \leq c_{16}e^{x/2}$ for all $x \geq 0$, we see that

$$\begin{aligned} \mathbf{E}\Big[\Sigma_1\Big] &\leq c_{17}\mathbf{E}\Big[\sum_{n\geq 0}\mathbf{E}\Big[\widetilde{Y}\Big(\mathrm{e}^{-\zeta_n/2}\widetilde{Y}\wedge 1\Big)\Big|\zeta_n\Big]\Big] \\ &= c_{17}\sum_{n\geq 0}\mathbf{E}\Big[\Big(\widetilde{Y}\Big)^2\mathrm{e}^{-\zeta_n}1_{\{\widetilde{Y}\leq \mathrm{e}^{\zeta_n/2}\}} + \widetilde{Y}1_{\{\widetilde{Y}>\mathrm{e}^{\zeta_n/2}\}}\Big] \\ &= c_{17}\mathbf{E}\bigg\{\Big(\widetilde{Y}\Big)^2\mathbf{E}\Big[\sum_{n\geq 0}\mathrm{e}^{-\zeta_n}1_{\{\zeta_n\geq 2\log\widetilde{Y}\}}\Big|\widetilde{Y}\Big] + \widetilde{Y}\mathbf{E}\Big[\sum_{n\geq 0}1_{\{\zeta_n<2\log\widetilde{Y}\}}\Big|\widetilde{Y}\Big]\bigg\}, \end{aligned}$$

where $\widetilde{Y}$ and (ζ_n) are independent. By (2.10),

$$\mathbf{E}\Big[\Sigma_1\Big] \;\leq\; c_{17}\mathbf{E}\Big[\Big(\widetilde{Y}\Big)^2\int_{2\log_+\widetilde{Y}}^{\infty} \mathrm{e}^{-x}R(x)U(\,\mathrm{d}x) + \widetilde{Y}\int_0^{2\log_+\widetilde{Y}} R(x)U(\,\mathrm{d}x)\Big], \tag{3.12}$$

which by (2.4) and (2.11) implies that

$$\mathbf{E}\Big[\Sigma_1\Big] \;\leq\; c_{17}c_2\mathbf{E}\Big[\Big(\widetilde{Y}\Big)^2\int_{2\log_+\widetilde{Y}}^{\infty} \mathrm{e}^{-x}(x+1)U(\,\mathrm{d}x) + \widetilde{Y}\int_0^{2\log_+\widetilde{Y}} (x+1)U(\,\mathrm{d}x)\Big] \tag{3.13}$$

$$\leq\; c_{18}\mathbf{E}\Big[\widetilde{Y}\Big(1+\log_+\widetilde{Y}\Big)^2\Big] = c_{18}\mathbf{E}\Big[Y\Big(1+\log_+ Y\Big)^2\Big] \tag{3.14}$$

On the other hand, in the same way, we obtain that

$$\mathbf{E}\Big[\Sigma_2\Big] \leq c_{19}\mathbf{E}\Big[Z\Big(1+\log_+ Z\Big)\Big]. \tag{3.15}$$

Consequently,

$$\mathbf{E}\Big[\Sigma_1\Big] + \mathbf{E}\Big[\Sigma_2\Big] \leq c_{20}\Big(\mathbf{E}\Big[Y+Z\Big] + \mathbf{E}\Big[Y\Big(\log_+ Y\Big)^2\Big] + \mathbf{E}\Big[Z\log_+ Z\Big]\Big). \tag{3.16}$$

Note that (3.1) ensures that $\mathbf{E}\Big[Y+Z\Big] < \infty$. The (3.11) is thus proved and we completes the proof of (3.6). $\square$

3.2 (1.8) is a necessary condition

This subsection is devoted to proving that

$$\max\Big\{\mathbf{E}\Big[Z\log_+ Z\Big],\ \mathbf{E}\Big[Y\Big(\log_+ Y\Big)^2\Big]\Big\} = \infty \Longrightarrow \mathbf{E}[D_\infty^{(0)}] = 0. \tag{3.17}$$

Proof of (3.17). According to (ii) of Theorem 3.1, we only need to show that

$$\forall y>0,\ \mathbf{P}\text{-a.s.}\ \sum_{n=1}^{\infty}\mathbf{E}_{\zeta_n}\Big[X;\ R(\zeta_n)\mathrm{e}^{-\zeta_n}X \geq y\Big] = \infty. \tag{3.18}$$

We break the assumption on the left-hand side of (3.17) up into three cases. In each case, we find out a different lower bound for X to establish (3.18). It hence follows that $D_\infty^{(0)}$ is trivial as $\mathbf{E}[D_\infty^{(0)}] = 0$. The three cases are stated as follows:

$$\mathbf{E}[Y(\log_+ Y)^2] = \infty, \qquad \mathbf{E}[Y(\log_+ Y)] < \infty; \tag{3.19a}$$

$$\mathbf{E}[Y(\log_+ Y)] = \infty; \tag{3.19b}$$

$$\mathbf{E}[Z(\log_+ Z)] = \infty. \tag{3.19c}$$

Proof of (3.18) under (3.19a) Recall that for any particle $x \in \mathbb{T} \setminus \{\varnothing\}$, $\Delta V(u) = V(u) - V(\overleftarrow{u})$, and that under $\mathbf{P}_a$, $(\Delta V(u);\ |u| = 1)$ is distributed as $\mathcal{L}$. For any $s \in \mathbb{R}$, we define a pair of random variables:

$$Y_+(s) := \sum_{|u|=1} \mathrm{e}^{-\Delta V(u)} 1_{(\Delta V(u) > -s)}, \qquad Y_-(s) := \sum_{|u|=1} \mathrm{e}^{-\Delta V(u)} 1_{(\Delta V(u) \le -s)}. \tag{3.20}$$

Clearly, $\widetilde{Y} = Y_+(s) + Y_-(s)$.

It follows from (3.9) and (2.4) that under $\mathbf{P}_{\zeta_n}$,

$$\begin{aligned} X &\geq \frac{\sum_{|u|=1} c_1(1 + \zeta_n + \Delta V(u))\mathrm{e}^{-\Delta V(u)} 1_{(\Delta V(u) > -\zeta_n/2)}}{c_2(1+\zeta_n)}, \\ &\geq \frac{\sum_{|u|=1} c_1(1/2 + \zeta_n/2)\mathrm{e}^{-\Delta V(u)} 1_{(\Delta V(u) > -\zeta_n/2)}}{c_2(1+\zeta_n)} \geq c_{21} Y_+(\zeta_n/2), \end{aligned}$$

where $\Big\{\Big(Y_+(s),\, Y_-(s)\Big);\ s \in \mathbb{R}\Big\}$ is independent of ζ_n and $c_{21} := \frac{c_1}{2c_2} > 0$. We thus see that the assertion that for any $y > 0$,

$$\sum_{n=1}^{\infty} \mathbf{E}\Big[Y_+(\zeta_n/2);\ R(\zeta_n/2)\mathrm{e}^{-\zeta_n} Y_+(\zeta_n/2) \geq y \Big| \zeta_n\Big] = \infty, \qquad \mathbf{P}\text{-a.s.}, \tag{3.21}$$

yields (3.18). It is known that $\zeta_n \to \infty$ as n goes to infinity (see, for example, [4]). It suffices that

$$\sum_{n=1}^{\infty} F(\zeta_n/2, \zeta_n) = \infty, \qquad \mathbf{P}\text{-a.s.} \tag{3.22}$$

where

$$F(s,z) := \mathbf{E}\Big[Y_+(s);\ \log Y_+(s) \geq z\Big], \quad s, z \in \mathbb{R}. \tag{3.23}$$

Let $F_1(z) := \mathbf{E}[Y;\ \log Y \geq z]$ which is positive and non-increasing. It follows from Lemma 2.1 and (3.1) that $\mathbf{E}[Y] = 1$. Therefore, for any $s,\ z \in \mathbb{R}$,

$$0 \leq F(s,z) \leq F_1(z) \leq \mathbf{E}[Y] = 1. \tag{3.24}$$

On the one hand, we deduce from (3.19a) that

$$\begin{aligned} \int_0^{\infty} F_1(y) y \,\mathrm{d}y &= \int_0^{\infty} \mathbf{E}\Big[Y 1_{(\log Y \geq y)}\Big] y \,\mathrm{d}y = \mathbf{E}\Big[Y \int_0^{(\log_+ Y)} y \,\mathrm{d}y;\ Y \geq 1\Big] \\ &= \mathbf{E}\Big[Y(\log_+ Y)^2\Big]/2 = \infty. \end{aligned}$$

According to Proposition 2.6, **P**-almost surely,

$$\sum_{n=1}^{\infty} F_1(\zeta_n) = \infty. \tag{3.25}$$

On the other hand, we can prove that $\sum_{n=1}^{\infty}\big[F_1(\zeta_n) - F(\zeta_n/2, \zeta_n)\big] < \infty$, **P**-a.s. In fact, as $Y = Y_+(s) + Y_-(s)$ under **P**, for any s, $y \in \mathbb{R}$,

$$\begin{aligned} F_1(y) - F(s,y) &= \mathbf{E}\Big[Y 1_{(\log Y \geq y)} - Y_+(s) 1_{(\log Y_+(s) \geq y)}\Big] \\ &= \mathbf{E}\Big[Y 1_{(\log Y \geq y > \log Y_+(s))} + Y 1_{(\log Y_+(s) \geq y)} - Y_+(s) 1_{(\log Y_+(s) \geq y)}\Big] \\ &= \mathbf{E}\Big[Y 1_{(\log Y \geq y > \log Y_+(s))} + Y_-(s) 1_{(\log Y_+(s) \geq y)}\Big]. \end{aligned}$$

Note that $Y \leq 2\max\{Y_+(s), Y_-(s)\}$ under **P**. It follows that

$$\begin{aligned} F_1(y) - F(s,y) &\leq \mathbf{E}\Big[2Y_-(s) 1_{(\log Y \geq y > \log Y_+(s),\, Y_+(s) \leq Y_-(s))} + Y 1_{(\log Y \geq y > \log Y_+(s),\, Y_+(s) > Y_-(s))}\Big] \\ &\quad + \mathbf{E}\Big[Y_-(s) 1_{(\log Y_-(s) \geq y)}\Big] \\ &\leq 3\mathbf{E}\Big[Y_-(s)\Big] + \mathbf{E}\Big[Y 1_{(\log Y \geq y > \log Y_+(s),\, Y_+(s) > Y_-(s))}\Big] \\ &\leq 3\mathbf{E}\Big[Y_-(s)\Big] + \mathbf{E}\Big[Y 1_{(\log Y \geq y > \log(Y/2))}\Big] =: d_1(s) + d_2(y). \end{aligned}$$

As a consequence,

$$\sum_{n=1}^{\infty}\big[F_1(\zeta_n) - F(\zeta_n/2, \zeta_n)\big] \leq \sum_{n\geq 0} d_1(\zeta_n/2) + \sum_{n \geq 0} d_2(\zeta_n). \tag{3.26}$$

Taking expectation on both sides yields that

$$\begin{aligned} \mathbf{E}\Big[\sum_{n=1}^{\infty}\big(F_1(\zeta_n) - F(\zeta_n/2, \zeta_n)\big)\Big] &\leq \mathbf{E}\Big[\sum_{n\geq 0} d_1(\zeta_n/2)\Big] + \mathbf{E}\Big[\sum_{n\geq 0} d_2(\zeta_n)\Big] \\ &= \int_0^{\infty} d_1(x/2) R(x) U(\,\mathrm{d}x) + \int_0^{\infty} d_2(x) R(x) U(\,\mathrm{d}x), \end{aligned} \tag{3.27}$$

where the last equality comes from (2.10).

For the first integration, we deduce from Lemma 2.1 that

$$d_1(s) = 3\mathbf{E}\Big[Y_-(s)\Big] = 3\mathbf{E}\Big[\sum_{|x|=1} \mathrm{e}^{-V(x)} 1_{(V(x) \leq -s)}\Big] = 3\mathbf{P}(-S_1 \geq s). \tag{3.28}$$

By (2.4), (2.11) and (3.1),

$$\begin{aligned}\int_0^\infty d_1(x/2)R(x)U(\mathrm{d}x) &= 3\int_0^\infty \mathbf{P}(-2S_1\ge x)R(x)U(\mathrm{d}x)\\ &= 3\mathbf{E}\Big[\int_0^{-2S_1} R(x)U(\mathrm{d}x); -2S_1\ge 0\Big]\\ &\le \le c_{22}\mathbf{E}\Big[\Big(1+(-2S_1)_+\Big)^2\Big]<\infty.\end{aligned}$$

For the second integration on the right-hand side of (3.27), as $d_2(y)=\mathbf{E}\Big[Y1_{(\log Y\ge y>\log(Y/2))}\Big]$, we use (2.4), (2.11) and (3.19a) to obtain that

$$\begin{aligned}\int_0^\infty d_2(x)R(x)U(\mathrm{d}x) &= \int_0^\infty \mathbf{E}\Big[Y1_{(\log Y\ge x>\log(Y/2))}\Big]R(x)U(\mathrm{d}x)\\ &= \mathbf{E}\Big[Y\int_{(\log Y-\log 2)_+}^{\log_+ Y} R(x)U(\mathrm{d}x)\Big]\\ &\le c_{23}\mathbf{E}\Big[Y(1+\log_+ Y)\Big]<\infty.\end{aligned}$$

Going back to (3.27), we conclude that

$$\mathbf{E}\Big[\sum_{n=1}^\infty \big(F_1(\zeta_n)-F(\zeta_n/2,\zeta_n)\big)\Big]\le \mathbf{E}\Big[\sum_{n\ge 0} d_1(\zeta_n/2)\Big]+\mathbf{E}\Big[\sum_{n\ge 0} d_2(\zeta_n)\Big]<\infty. \tag{3.29}$$

Therefore, $\mathbf{P}$-a.s.,

$$\sum_{n=1}^\infty \big[F_1(\zeta_n)-F(\zeta_n/2,\zeta_n)\big]<\infty, \tag{3.30}$$

which, combined with (3.25), implies (3.22). Thus (3.18) is proved under (3.19a).

Proof of (3.18) under (3.19b) Now we suppose that $\mathbf{E}[Y\log_+ Y]=\infty$. By (2.4), we observe that under $\mathbf{P}_{\zeta_n}$,

$$\begin{aligned}X &= \frac{\sum_{|u|=1} R(\Delta V(u)+\zeta_n)\mathrm{e}^{-\Delta V(u)}1_{(\Delta V(u)>-\zeta_n)}}{R(\zeta_n)}\\ &\ge c_1\frac{Y_+(\zeta_n)}{R(\zeta_n)},\end{aligned} \tag{3.31}$$

where $\{Y_+(s);\ s\in\mathbb{R}\}$ and ζ_n are independent.

To establish (3.18), we only need to show that for any $y\ge 1$,

$$\sum_{n\ge 1}\mathbf{E}\Big[\frac{Y_+(\zeta_n)}{R(\zeta_n)};\ Y_+(\zeta_n)\ge ye^{\zeta_n}\Big|\zeta_n\Big]=\sum_{n\ge 1}\frac{F(\zeta_n,\log y+\zeta_n)}{R(\zeta_n)}=\infty,\qquad \mathbf{P}\text{-a.s.} \tag{3.32}$$

For any $y \geq 1$ fixed, let

$$F_2(x) := \frac{F_1(\log y + x)}{R(x)}, \ \forall x \geq 0, \tag{3.33}$$

which is non-increasing as $R(x) = U^-([0, x))$ is non-decreasing and F_1 is non-increasing. One sees that

$$\sum_{n\geq 1} F_2(\zeta_n) = \sum_{n\geq 1} \frac{F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)} + \sum_{n\geq 1} \frac{F_1(\log y + \zeta_n) - F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)}. \tag{3.34}$$

By (2.4), $\frac{F_1(\log y+x)}{c_2(1+x)} \leq F_2(x) \leq \frac{1}{c_1}$. It then follows from (3.19b) that

$$\begin{aligned}
\int_0^\infty F_2(x) x \,\mathrm{d}x &\geq \int_0^\infty F_1(\log y + x) \frac{x}{c_2(1+x)} \,\mathrm{d}x \\
&\geq \int_1^\infty c_{24} \mathbf{E}\Big[Y 1_{(\log Y \geq \log y + x)}\Big] \,\mathrm{d}x \\
&\geq c_{24} \mathbf{E}[Y(\log Y - \log y - 1)_+] = \infty.
\end{aligned}$$

By Proposition 2.6,

$$\sum_{n\geq 0} F_2(\zeta_n) = \sum_{n\geq 0} \frac{F_1(\log y + \zeta_n)}{R(\zeta_n)} = \infty, \ \mathbf{P}\text{-a.s.} \tag{3.35}$$

In view of (3.34) and (3.35), it suffices to show that $\mathbf{P}$-a.s.,

$$\sum_{n\geq 0} \frac{F_1(\log y + \zeta_n) - F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)} < \infty. \tag{3.36}$$

Recall that $F_1(z) - F(s, z) \leq d_1(s) + d_2(z)$. By (2.10),

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{n\geq 0} \frac{F_1(\log y + \zeta_n) - F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)}\Big] \\
\leq\ &\mathbf{E}\Big[\sum_{n\geq 0} \frac{d_1(\zeta_n) + d_2(\log y + \zeta_n)}{R(\zeta_n)}\Big] = \int_0^\infty \Big[d_1(x) + d_2(\log y + x)\Big] U(\,\mathrm{d}x).
\end{aligned} \tag{3.37}$$

On the one hand, recalling that $d_1(x) = 3\mathbf{P}(-S_1 \geq x)$, we deduce from (2.11) that

$$\begin{aligned}
\int_0^\infty d_1(x) U(dx) &= \int_0^\infty 3\mathbf{P}(-S_1 \geq x) U(dx) \\
&= 3\mathbf{E}\Big[\int_0^{(-S_1)_+} U(dx)\Big] \\
&\leq 3c_4 \mathbf{E}\Big[1 + (-S_1)_+\Big] < \infty.
\end{aligned} \tag{3.38}$$

On the other hand, recalling that $d_2(x) = \mathbf{E}[Y;\ \log Y \geq x > \log Y - \log 2]$, by (2.11) again, we obtain that

$$
\begin{aligned}
(3.39) \qquad \int_0^\infty d_2(\log y + x) U(dx) &= \int_0^\infty \mathbf{E}[Y 1_{(\log Y \geq \log y + x > \log Y - \log 2)}] U(dx) \\
&= \mathbf{E}\Big[Y \int_{(\log Y - \log y - \log 2)_+}^{(\log Y - \log y)_+} U(dx)\Big] \\
&\leq c_4(1 + \log 2)\mathbf{E}[Y] < \infty.
\end{aligned}
$$

Combined with (3.38) and (3.39), (3.37) becomes that

$$
(3.40) \qquad \mathbf{E}\Big[\sum_{n\geq 1} \frac{F_1(\log y + \zeta_n) - F(\zeta_n, \log y + \zeta_n)}{R(\zeta_n)}\Big] < \infty.
$$

We thus get (3.36), and completes the proof of (3.18) given (3.19b).

Proof of (3.18) under (3.19c) In this part we assume that $\mathbf{E}[Z \log_+ Z] = \infty$ with $Z = \sum_{|u|=1} V(u)_+ \mathrm{e}^{-V(x)} \geq 0$. We observe that under $\mathbf{P}_{\zeta_n}$,

$$
\begin{aligned}
X &\geq \frac{\sum_{|u|=1} R(\Delta V(u) + \zeta_n) \mathrm{e}^{-\Delta V(u)} 1_{(\Delta V(u) > 0)}}{R(\zeta_n)} \\
(3.41) \qquad &\geq \frac{c_1}{R(\zeta_n)} \widetilde{Z},
\end{aligned}
$$

where $\widetilde{Z} = \sum_{|x|=1} \Big(\Delta V(x)\Big)_+ \mathrm{e}^{-\Delta V(x)}$ is independent of ζ_n. As a consequence, for any $y > 0$,

$$
(3.42) \qquad \sum_{n\geq 1} \mathbf{E}_{\zeta_n}\Big[X;\ R(\zeta_n)\mathrm{e}^{-\zeta_n} X \geq y\Big] \geq \sum_{n\geq 1} \frac{c_1}{R(\zeta_n)} \mathbf{E}\Big[\widetilde{Z};\ c_1 \widetilde{Z} \geq y \mathrm{e}^{\zeta_n} \Big| \zeta_n\Big].
$$

Recall that $\widetilde{Z}$ is distributed as Z under $\mathbf{P}$. Therefore, it is sufficient to prove that for any $y > 0$,

$$
(3.43) \qquad \sum_{n\geq 1} \frac{1}{R(\zeta_n)} \mathbf{E}\Big[\widetilde{Z};\ \widetilde{Z} \geq y \mathrm{e}^{\zeta_n} \Big| \zeta_n\Big] = \sum_{n\geq 1} F_3(\zeta_n) = \infty, \qquad \mathbf{P}\text{-a.s.}
$$

where

$$
(3.44) \qquad F_3(z) := \frac{\mathbf{E}[Z;\ \log Z \geq z + \log y]}{R(z)}, \ \forall z \geq 0.
$$

Since R is non-decreasing, the function F_3 is non-increasing. By Lemma 2.1 and (2.4),

$$
(3.45) \qquad 0 \leq F_3(z) \leq \frac{\mathbf{E}[Z]}{R(z)} \leq \frac{\mathbf{E}[(S_1)_+]}{c_1} < \infty.
$$

Moreover, by (2.11) and (3.19c),

$$\int_0^\infty F_3(x)x\,\mathrm{d}x \ \geq\ \int_1^\infty c_{25}\mathbf{E}\Big[Z;\ \log Z - \log y \geq x\Big]\,\mathrm{d}x \qquad (3.46)$$
$$\geq\ c_{25}\mathbf{E}\Big[Z(\log Z - \log y - 1)_+\Big] = \infty.$$

Because of Proposition 2.6, we obtain that for any $y > 0$,

$$\sum_{n\geq 1}\frac{1}{R(\zeta_n)}\mathbf{E}\Big[\widetilde{Z};\ \widetilde{Z} \geq y\mathrm{e}^{\zeta_n}\Big|\zeta_n\Big] = \sum_{n\geq 1}F_3(\zeta_n) = \infty, \qquad \mathbf{P}\text{-a.s.} \qquad (3.47)$$

which completes the proof of (3.18) under (3.19c).

□

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